数理解析研究所講究録
第 1671 巻 2009 年 146-155



# Existence of maximizers for functionals of critical growth

石渡 通徳
Michinori Ishiwata *
Muroran Institute of Technology, Muroran 050-8585, Japan

## 1 Introduction and main results

The classical Trudinger-Moser inequality asserts that, for $\alpha\in(0,\alpha_N]$, there exists $B_{N,\alpha}>0$ which depends only on $N$ $(N\geq 2)$ and $\alpha$ satisfying

$$\int_{\Omega}e^{\alpha|u|^{\frac{N}{N-1}}}\leq B_{N,\alpha}|\Omega| \tag{1}$$

for all bounded $\Omega\subset\mathbb{R}^N$ and for $u\in W_0^{1,N}(\Omega)$ with $\|\nabla u\|_{L^N(\Omega)}=1$, where $\alpha_N:=N|S^{N-1}|^{\frac{1}{N-1}}$ and $|S^{N-1}|$ is the surface area of the $(N-1)$-dimensional unit sphere, see [14, 11]. Let

$$b_{N,\alpha}:=\sup_{u\in W_0^{1,N},\ \|\nabla u\|_N=1}\frac{\int_{\Omega}e^{\alpha|u|^{\frac{N}{N-1}}}}{|\Omega|}.$$

The existence of a maximizer associated with $b_{N,\alpha}$ is shown by Carleson-Chang in [5] when $\Omega$ is an $N$-dimensional ball and by Flucher [6] when $\Omega$ is a general bounded domain in $\mathbb{R}^2$.

There is an extension of this inequality to unbounded domains. Let $N\geq 2$, $\alpha\in(0,\alpha_N]$ and let

$$\Phi_{N,\alpha}(t)=e^{\alpha t}-\sum_{j=0}^{N-2}\frac{\alpha^j}{j!}t^j.$$

*This research was partially supported by the Grant-in-Aid for Young Scientists B #19740081, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

It is known that there exists $D_{N,\alpha}$ which only depends on $N$ and $\alpha$ satisfying

$$\int_{\mathbb{R}^N} \Phi_{N,\alpha}(u^{\frac{N}{N-1}}) \leq D_{N,\alpha} \tag{2}$$

for all $u \in W_0^{1,N}(\mathbb{R}^N)$ with $\|u\|_{W^{1,N}(\mathbb{R}^N)} = 1$. The inequality (2) with $N = 2$ is introduced by Cao [4]. Later B. Ruf proved in [13] that $\alpha_2 = 4\pi$ is a critical exponent. The case $N \geq 3$ is also treated in a recent paper [10].

The purpose of this note is to show the attainability of the best constant $d_{N,\alpha}(\mathbb{R}^N)$ associated with (2), where

$$d_{N,\alpha}(\mathbb{R}^N) := \sup_{u \in W^{1,N}(\mathbb{R}^N),\ \|u\|_{W^{1,N}(\mathbb{R}^N)}=1} \int_{\mathbb{R}^N} \Phi_{N,\alpha}(u^{\frac{N}{N-1}}).$$

In [10], Li-Ruf proved that $d_{N,\alpha}$ with $N \geq 3$ and with $\alpha = \alpha_N$ (critical case) is attained. The method used in [10] is a blow-up technique and cannot be applied to the $N = 2$ case. The two dimensional case with $\alpha = \alpha_2 = 4\pi$ is treated by Ruf in [13] and it is claimed that $d_{2,4\pi}$ is attained. In the present note, we treat the subcritical case and the critical case in a unified way based on the concentration-compactness type argument [8, 9, 3, 2]. Moreover, we also obtain the nonexistence result of maximizers for $d_{2,\alpha}$ with small $\alpha$.

Our main results read as follows.

**Theorem 1.1**

*Let $N \geq 2$ and let $\alpha_N = N|S^{N-1}|^{\frac{1}{N-1}}$, where $|S^{N-1}|$ is the surface area of the $(N-1)$-dimensional unit sphere. Also, let*

$$B_2 = \sup_{\phi \neq 0, \phi \in H^1} \frac{\|\phi\|_4^4}{\|\nabla \phi\|_2^2 \|\phi\|_2^2} \tag{3}$$

*if $N = 2$. Then $d_{N,\alpha}(\mathbb{R}^N)$ is attained for $0 < \alpha < \alpha_N$ with $N \geq 3$ and for $2/B_2 < \alpha \leq \alpha_2 = 4\pi$ with $N = 2$.*

The number $B_2$ is the best constant of the (two-dimensional) $H^1$-Moser inequality

$$\|\phi\|_4^4 \leq B_2 \|\nabla \phi\|_2^2 \|\phi\|_2^2, \quad \phi \in H^1,\ \phi \neq 0.$$

It is known that the interval $(2/B_2, \alpha_2]$ is non-empty, see e.g. [15, 1].

**Theorem 1.2**
*Let $N = 2$. If $\alpha \ll 1$, then $d_{2,\alpha}(\mathbb{R}^2)$ is not attained.*

For the variational problem associated with (2), it is enough to consider radially symmetric nonnegative functions by virtue of symmetrization. Hence, in the following, we only consider radially symmetric, nonnegative functions.

**Notation** $\|\cdot\|_{L^p(\Omega)}$ denotes the standard $L^p(\Omega)$-norm. We occasionally omit the subscript $\Omega$ and we also use the abbreviation $\|\cdot\|_p$. The norm of $W^{1,N}(\Omega)$ is defined by $\|u\|^N_{W^{1,N}(\Omega)} := \|\nabla u\|^N_{L^N(\Omega)} + \|u\|^N_{L^N(\Omega)}$. $B_R$ denotes the ball in $\mathbb{R}^N$ with radius $R$ centered at the origin and $B^c_R$ its complement. $\mathcal{M}(\Omega)$ is a set consists of Radon measures in $\Omega$. $W^{1,N}_r$ denotes the set consists of radially symmetric $W^{1,N}$-functions. $|B^N|$ and $|S^{N-1}|$ denote the volume of the $N$-dimensional unit ball and the surface area of the $(N-1)$-dimensional unit sphere, respectively. Let $\alpha_N := N|S^{N-1}|^{\frac{1}{N-1}}$. The constant $C$ may vary from line to line. We pass to subsequences freely.

# 2 Proof of Theorem 1.1

The proof of Theorem 1.1 needs the study of the supremum of the value $\int \Phi_{N,\alpha}(u_n^{\frac{N}{N-1}})$ with vanishing or concentrating sequence $(u_n)$. At first we introduce the definition of a vanishing/concentrating sequence. Let us introduce the following quantities which measure the lack of mass:

$$\mu_0 = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B_R} (|\nabla u_n|^N + |u_n|^N), \tag{4}$$

$$\mu_\infty = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B^c_R} (|\nabla u_n|^N + |u_n|^N), \tag{5}$$

$$\nu_0 = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B_R} \Phi_{N,\alpha}(u_n^{\frac{N}{N-1}}), \tag{6}$$

$$\nu_\infty = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B^c_R} \Phi_{N,\alpha}(u_n^{\frac{N}{N-1}}), \tag{7}$$

$$\eta_0 = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B_R} |u_n|^N, \ \eta_\infty = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B^c_R} |u_n|^N. \tag{8}$$

**Definition 2.1**

*Let $(u_n) \subset W^{1,N}$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W^{1,N}$.*
(a) *It is said that $(u_n)$ is a normalized concentrating sequence ((NCS) in short) if $(u_n)$ satisfies $\|u_n\|_{W^{1,N}} = 1$, $u = 0$ and $\lim_{n\to\infty} \int_{B_\rho^c} |\nabla u_n|^N + |u_n|^N = o(1)$ for all $\rho > 0$.*
(b) *It is said that $(u_n)$ is a normalized vanishing sequence ((NVS) in short) if $(u_n)$ satisfies $\|u_n\|_{W^{1,N}} = 1$, $u = 0$ and $\nu_0 = 0$, where $\nu_0$ is defined by (6).*

Next we introduce the obstacle values for the compactness of maximizing sequences.

**Definition 2.2**

(a) *A number*

$$d_{\rm ncl}(N,\alpha) = \sup\{c; \text{there exists a (NCS) } (u_n) \text{ s.t. } c = \limsup_{n\to\infty} \int \Phi_{N,\alpha}(u_n^{\frac{N}{N-1}})\}$$

*is called a normalized concentration limit.*
(a) *A number*

$$d_{\rm nvl}(N,\alpha) = \sup\{c; \text{there exists a (NVS) } (u_n) \text{ s.t. } c = \limsup_{n\to\infty} \int \Phi_{N,\alpha}(u_n^{\frac{N}{N-1}})\}.$$

*is called a normalized vanishing limit.*

Ruf proved in [13] that

$$d_{\rm ncl}(2,\alpha_2) = e\pi. \tag{9}$$

Moreover, we can show the following:

**Proposition 2.1**

*Let $N \geq 2$.*
(a) *Let $\alpha \in (0,\alpha_N]$. Then there holds $d_{\rm nvl}(N,\alpha) = \frac{\alpha^{N-1}}{(N-1)!}$.*
(b) *It holds that $d_{N,\alpha} > \frac{\alpha^{N-1}}{(N-1)!}$ for $\alpha \in (0,\alpha_N]$ if $N \geq 3$ and for $\alpha \in (2/B_2, \alpha_2]$ if $N = 2$, where $B_2$ is the best constant of $H^1$-Moser inequality defined by (3).*

Let $N = 2$. From Proposition 2.1 and (9), we see that

$$d_{2,\alpha_2} > d_{\rm nvl}(2, \alpha_2) = \alpha_2 = 4\pi > e\pi = d_{\rm ncl}(2, \alpha_2).$$

From this relation, it is observed that the main obstacle to the compactness of the maximizing sequences for $d_{2,\alpha_2}$ is not the concentrating behavior but the vanishing behavior. Hence the exclusion of the vanishing behavior of maximizing sequences is crucial for the verification of a maximizer associated with $d_{2,\alpha_2}$ and this analysis is not given in [13].

**Sketch of the proof of Theorem 1.1** Let $\alpha \in (0, \alpha_N)$ if $N \geq 3$ and let $\alpha \in (2/B_2, \alpha_2]$ if $N = 2$. Also let $(u_n)$ be a maximizing sequence for $d_{N,\alpha}$. By virtue of the radially symmetric rearrangement, we can assume that $u_n$ is a radially symmetric, nonnegative function which is decreasing in the radial coordinate. Since $\|u_n\|_{W^{1,N}} = 1$, we can find $u \in W^{1,N}$ such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,N}. \tag{10}$$

Let $\phi_R^0$, $\phi_R^\infty \in C_0^\infty$ be cut-off functions satisfying

$$0 \leq \phi_R^0 \leq 1, \quad \phi_R^0 = 1 \text{ in } B_R, \quad \phi_R^0 = 0 \text{ in } B_{R+1}^c, \tag{11}$$

and

$$0 \leq \phi_R^\infty \leq 1, \quad \phi_R^\infty = 0 \text{ in } B_R, \quad \phi_R^\infty = 1 \text{ in } B_{R+1}^c, \tag{12}$$

respectively. Also let $u_{n,R}^* := u_n \phi_R^*$, where $* = 0, \infty$. By direct computations, we can show

$$1 = \mu_0 + \mu_\infty, \quad 1 \geq \eta_0 + \eta_\infty \quad \text{and} \quad d_{N,\alpha} = \nu_0 + \nu_\infty. \tag{13}$$

Moreover, the concentration-compactness type argument as in [2, 3, 7] yields the following alternative.

**Lemma 2.1**
*It holds either*

$$(\mu_0, \nu_0) = (1, d_{N,\alpha}) \text{ and } (\mu_\infty, \nu_\infty) = (0, 0) \tag{14}$$

*or*

$$(\mu_0, \nu_0) = (0, 0) \text{ and } (\mu_\infty, \nu_\infty) = (1, d_{N,\alpha}). \tag{15}$$

Now we can show that vanishing cannot occur for maximizing sequences:

**Proposition 2.2**
*It holds that*

$$(\mu_0, \nu_0) = (1, d_{N,\alpha}) \text{ and } (\mu_\infty, \nu_\infty) = (0,0). \tag{16}$$

**Proof of Proposition 2.2.**
We show that (15) cannot occur. Indeed, assume that (15) is true. Note that in this case, $(u_n)$ is a normalized vanishing sequence. Therefore, by Proposition 2.1 (a), we have

$$d_{N,\alpha} = \nu_\infty = \limsup_{R\to\infty} \limsup_{n\to\infty} \int_{B_R^c} \Phi_{N,\alpha}(u_n^{\frac{N}{N-1}}) \le d_{\mathrm{nvl}} = \frac{\alpha^{N-1}}{(N-1)!},$$

which contradicts Proposition 2.1 (b). Consequently, the only possible case is (14) and this completes the proof. ∎

**Proposition 2.3**
*It holds that $u \neq 0$.*

**Proof of Proposition 2.3.**
We only treat the case $N = 2$ and $\alpha = \alpha_2$, since the other case is rather easy by virtue of the local compactness. Assume that the conclusion is not true and that $u = 0$. We first show that, under this assumption, $(u_{n,R}^0)$ is a (NCS). To this end, it is enough to verify that

$$\int_{B_\rho^c} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2) \to 0 \tag{17}$$

as $n \to \infty$ for any $\rho > 0$. Let $w_{n,R} := \frac{u_{n,R}^0}{\|\nabla u_{n,R}^0\|_2}$. Since $\mu_\infty = 0$ and $u_{n,R}^0 \to 0$ strongly in $L^2$ by the assumption, we see that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u_{n,R}^0|^2 \ge \frac{1}{2} \tag{18}$$

for large $R$. Fix such $R > 0$. Note that $u_{n.R}^0 \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^2)$. Thus by the concentration-compactness lemma [8, 9], we obtain

$$|\nabla w_{n,R}|^2 \rightharpoonup \delta_0 \text{ weakly in } \mathcal{M}(\mathbb{R}^2), \quad w_{n,R} \to 0 \text{ strongly in } L^2 \tag{19}$$

as $n \to \infty$.

Let $\phi_{\rho,R}$ be a smooth cut-off function satisfying $0 \le \phi_{\rho,R} \le 1$ in $\mathbb{R}^2$, $\phi_{\rho,R} = 1$ in $B_R \setminus B_\rho$ and $\phi_{\rho,R} = 0$ in $B_{\rho/2} \cup B_{R+1}^c$. Then

$$\int_{B_\rho^c} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2) = \int_{B_\rho^c \cap B_R} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2) + \int_{B_\rho^c \cap B_R^c} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2).$$

By (18) and (19), we obtain

$$\begin{aligned}
&\int_{B_\rho^c \cap B_R} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2) \le \int_{B_\rho^c \cap B_R} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2)\phi_{\rho,R} + o(1) \\
&\le \|\nabla u_{n,R}^0\|_2^2 \int_{B_\rho^c \cap B_R} (|\nabla u_{n,R}^0|^2 + |u_{n,R}^0|^2)\phi_{\rho,R} + o(1) \le C\langle \delta_0, \phi_{\rho,R}\rangle + o(1) \\
&= C\phi_{\rho,R}(0) + o(1) = o(1)
\end{aligned}$$

as $n \to \infty$. This relation gives (17). Combining this fact with $\mu_\infty = 0$, we also see that $(u_n)$ is a (NCS). Then by using (9), we have

$$d_{2,\alpha_2} = \lim_{n\to\infty} \int \Phi_{2,\alpha}(u_n^2) \le d_{\rm ncl}(2, \mathbb{R}^2) = e\pi,$$

which contradicts Proposition 2.1 (b). Hence we have $u \neq 0$. ∎

Now the verification of the fact that $u$ is a maximizer is rather standard.

# 3 Proof of Theorem 1.2

In this section, we always assume that $N = 2$ and $\alpha < 4\pi$. Let $M := \{u \in H^1(\mathbb{R}^2); \|u\|_{H^1(\mathbb{R}^2)} = 1\}$. For any $v \in M$, we introduce the following family of functions $v_t$ given by

$$v_t(x) := \sqrt{t}v(\sqrt{t}x),$$

where $t > 0$ is a positive parameter. Let $w_t := v_t/\|v_t\|_{H^1(\mathbb{R}^2)}$. Then $w_t$ is a curve in $M$ passing through $v$, since $\|w_t\|_{H^1(\mathbb{R}^2)} = 1$ and $w_1 = v_1/\|v_1\|_{H^1} = v/\|v\|_{H^1} = v$. Therefore, if $v$ is a critical point of $J_{2,\alpha}(u) := \int_{\mathbb{R}^2} \Phi_{2,\alpha}(u^2)$, then

$$\frac{d}{dt} J_{2,\alpha}(w_t)\bigg|_{t=1} = 0. \tag{20}$$

Now we compute the left hand side of (20). By using the fact $\|v_t\|_p^p = t^{(p-2)/2}\|v\|_p^p$ and $\|\nabla v_t\|_2^2 = t^4\|\nabla v\|_2^2$, we see that

$$J_{2,\alpha}(w_t) = J_{2,\alpha}\left(\frac{v_t}{\|v_t\|_{H^1(\mathbb{R}^2)}}\right) = \sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\frac{t^{j-1}\|v\|_{2j}^{2j}}{(t\|\nabla v\|_2^2+\|v\|_2^2)^j}.$$

Hence, in view of $\|v\|_{H^1} = 1$, we have

$$\begin{aligned}\left.\frac{d}{dt}J_{2,\alpha}(w_t)\right|_{t=1} &= \left.\sum_{j=1}^{\infty}\frac{\alpha^j}{j!}\frac{t^{j-2}\|v\|_{2j}^{2j}}{(t\|\nabla v\|_2^2+\|v\|_2^2)^{j+1}}(-t\|\nabla v\|_2^2+(j-1)\|v\|_2^2)\right|_{t=1}\\ &\le -\alpha\|v\|_2^2\|\nabla v\|_2^2+\sum_{j=2}^{\infty}\frac{\alpha^j}{(j-1)!}\|v\|_{2j}^{2j}\\ &= \alpha\|v\|_2^2\|\nabla v\|_2^2\left[-1+\sum_{j=2}^{\infty}\frac{\alpha^{j-1}}{(j-1)!}\frac{\|v\|_{2j}^{2j}}{\|v\|_2^2\|\nabla v\|_2^2}\right]. \qquad (21)\end{aligned}$$

Here we take any $\beta \in (\alpha, 4\pi)$. By using the Gagliardo-Sobolev-Nirenberg inequality with the sharp asymptotics (see e.g.[12]), we have

$$\frac{\|v\|_{2j}^{2j}}{\|v\|_2^2\|\nabla v\|_2^2} \le C_\beta\frac{j!}{\beta^j},$$

where $C_\beta$ is a constant only depend on $\beta$. From this relation, we see that

$$\begin{aligned}(21) &\le \alpha\|v\|_2^2\|\nabla v\|_2^2\left[-1+\alpha C_\beta\sum_{j=2}^{\infty}\frac{\alpha^{j-2}}{\beta^j}j\right]\\ &\le \alpha\|v\|_2^2\|\nabla v\|_2^2\left[-1+\alpha C_\beta C\right], \qquad (22)\end{aligned}$$

where $C > 0$ is a constant independent of $\alpha$, $\beta$. Consequently, we have

$$\left.\frac{d}{dt}J_{2,\alpha}(w_t)\right|_{t=1} = \alpha\|v\|_2^2\|\nabla v\|_2^2\left[-1+\alpha C_\beta C\right] < 0$$

for $\alpha < 1/(C_\beta C)$. Hence, in view of (20), no $v$ can be a critical point of $J_{2,\alpha}$ in $M$ when $\alpha < 1/(C_\beta C)$. This completes the proof of Theorem 1.2.

**Acknowledgments.** The author would like to thank N. Ikoma for a stimulating discussion concerning Theorem 1.2.

# References

[1] Beckner, William, Estimates on Moser embedding. Potential Anal. 20 (2004), 345–359.

[2] Bianchi, Gabriele; Chabrowski, Jan; Szulkin, Andrzej, On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent. Nonlinear Anal. 25 (1995), no. 1, 41–59.

[3] Ben-Naoum, A. K.; Troestler, C.; Willem, M., Extrema problems with critical Sobolev exponents on unbounded domains. Nonlinear Anal. 26 (1996), no. 4, 823–833.

[4] Cao, D. M., Nontrivial solution of semilinear elliptic equation with critical exponent in $R^2$. Comm. Partial Differential Equations 17 (1992), no. 3-4, 407–435.

[5] Carleson, Lennart; Chang, Sun-Yung A., On the existence of an extremal function for an inequality of J. Moser. Bull. Sci. Math. (2) 110 (1986), no. 2, 113–127.

[6] Flucher, Martin, Extremal functions for the Trudinger-Moser inequality in 2 dimensions. Comment. Math. Helv. 67 (1992), no. 3, 471–497.

[7] Ishiwata, Michinori, Mitsuharu, Ôtani, Concentration compactness principle at infinity with partial symmetry and its application. Nonlinear Anal. 51 (2002), 391–407.

[8] Lions, P.-L. The concentration-compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1 (1985), no. 1, 145–201.

[9] Lions, P.-L. The concentration-compactness principle in the calculus of variations. The limit case. II. Rev. Mat. Iberoamericana 1 (1985), no. 2, 45–121.

[10] Li, Yuxiang; Ruf, Bernhard, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^n$. Indiana Univ. Math. J. 57 (2008), no. 1, 451–480.

[11] Moser, J. A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. 20 (1970/71), 1077–1092.

[12] Ogawa, Takayoshi, A proof of Trudinger's inequality and its application to nonlinear Schrodinger equations. Nonlinear Anal. 14 (1990), no. 9, 765–769.

[13] Ruf, Bernhard, A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$. J. Funct. Anal. 219 (2005), no. 2, 340–367.

[14] Trudinger, Neil S. On imbeddings into Orlicz spaces and some applications. J. Math. Mech. 17 (1967) 473–483.

[15] Weinstein, Michael I. Nonlinear Schrodinger equations and sharp interpolation estimates. Comm. Math. Phys. 87 (1982/83), no. 4, 567–576.